Topological Spaces

Defn. A **topology** on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X called **open sets** with

I. $\varnothing, X \in \mathcal{T}$
II. $\{U_\alpha\}_{\alpha\in\Lambda} \subset \mathcal{T} \implies \bigcup_{\alpha\in\Lambda} U_\alpha \in \mathcal{T}$
III. $U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T}$

Defn. A set $C \subset X$ is **closed** if its complement $X \setminus C \in \mathcal{T}$ is open. Closed sets satisfy

I. $\varnothing$ and X are closed.
II. finite unions of closed sets are closed
III. arbitrary intersections of closed sets are closed
IV. a subset $A \subset X$ is closed if and only if $\overline{A} = A$

Closure, Interior, and Boundary

The **closure** of a set $A \subset X$ is defined to be the smallest closed set $\overline{A}$ containing A, that is, the intersection of all closed sets containing A. Equivalently, $x \in \overline{A}$ if and only if A intersects every neighborhood of x. Informally, the closure contains all points beyond A which are *topologically indistinguishable* from A in the sense that they cannot be separated from A by an open set.

Property 1. $\overline{A} = \{x \in X \mid \forall x \in U_x \in \mathcal{T},\ A \cap U_x \neq \varnothing\}$.

Defn. The **interior** A° of $A \subset X$ is the union of all open subsets of A; the largest open subset of A.

I. $A^\circ \subset A \subset \overline{A}$.
II. A connected $\implies \overline{A}$ connected.
III. $x \in \overline{A}$ iff every open nbhd of x intersects A
IV. $\overline{A}$ contains all limit points of A

Defn. The **boundary** of $A \subset X$ is $\partial A \equiv \overline{A} \setminus A^\circ$.

Cluster Points

Defn. Let $(X, \mathcal{T})$ be a topological space. A point $x \in X$ is a **cluster point** of a subset $A \subset X$ iff every neighborhood of x intersects A at some point other than x itself, that is, $x \in \overline{A \setminus x}$.

Theorem 1. *Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. If A' is the set of cluster points of A, then $\overline{A} = A \cup A'$.*

Proof. We already have $A \subset \overline{A}$. If $x \in A'$ is a cluster point, then every neighborhood of x intersects A, so $x \in \overline{A}$ by definition. Conversely, if $x \in \overline{A}$, then either $x \in A$ or every neighborhood of x intersects A, so $x \in A'$. □

Corollary 1. *A subset of a topological space is closed if and only if it contains all its cluster points.*

- In a T_1 space, x is a cluster point of A if and only if every neighborhood of x contains infinitely many points of A

Continuity

DEFN. $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is **continuous** if preimages of open sets are open, that is,

$$\forall U \in \mathcal{T}_Y, f^{-1}(U) \in \mathcal{T}_X$$

I. Equivalently, preimages of closed sets are closed.
II. The composition of continuous functions is continuous.

Property 2. *Continuity preserves many topological properties:*

I. *The continuous image of a compact set is compact (Theorem 11).*
II. *The continuous image of a connected set is connected (Theorem 5).*

Theorem 2, (Pasting Lemma)**.** *Topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Suppose $X = A \cup B$ where $A, B \subset X$ are either both closed or both open. If $f : X \to Y$ is continuous when restricted to both A and B, then f is continuous.*

Proof. Assume $A, B \subset X$ are closed; the proof for open sets is similar. Let $C \subset Y$ be closed. Since $f|_A$ is continuous, the preimage $f|_A^{-1}(C)$ is closed in the subspace topology of A, therefore closed in X, and similarly for $f|_B^{-1}(C)$. Thus $f^{-1}(C) = f|_A^{-1}(C) \cup f|_B^{-1}(C)$ is closed in X, as desired. □

Homeomorphisms

Sequences

DEFN. A sequence $(x_n)_{n=1}^{\infty}$ in a topological space $(X, \mathcal{T})$ **converges** to $x \in X$ iff each open neighborhood U_x of x contains a tail of the sequence, that is, $(x_n)_{n \geq N} \subset U_x$ for some $N \in \mathbb{N}$.

I. If $x_n \to x$, then every subsequence $x_{n_k} \to x$.

EXAMPLE 1. Sequences do not necessarily have unique limits; moreover, subsequences of a convergent sequence may have limits that the parent sequence does not! Consider the topology $\mathcal{T} = \{b, ab, ac, bc, abc\}$ on three points a, b, c. TODO: When are these counterexamples avoided? Must we have unique limits or is some weaker condition enough? Then,

I. $b, b, b, b, \ldots$ converges to both b and a
II. $a, b, a, b, \ldots$ converges to only a, but the subsequence $b, b, b, \ldots$ also converges to b!

Sequential Spaces

DEFN. A set $U \subset X$ is **sequentially open** iff every sequence $x_n \in X$ converging to a point $x \in U$ eventually lies entirely within U, that is,

DEFN. A set $C \subset X$ is **sequentially closed** iff, whenever a sequence $x_n \in C$ converges to a point $x \in X$, then $x \in C$.

DEFN.

Property 3. *Easy properties of sequentially closed / open sets.*

I. *Every closed set is sequentially closed, in particular $\overline{A}$ is.*
II. *Every open set is sequentially open.*
III. *If $U \subset X$ is sequentially open, then $X \setminus A$ is sequentially closed.*
IV. *If $C \subset X$ is sequentially closed, then $X \setminus C$ is sequentially open.*

In general, sequential closed/openness is not equivalent to closed/openness.

EXAMPLE 2. A sequentially closed set is not necessarily closed.

Fréchet-Urysohn Spaces

Topological Subspaces

DEFN. The **subspace topology** of $Y \subset X$ wrt $(X, \mathcal{T}_X)$ is

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}_X\}$$

Theorem 3. *If $\mathcal{B}$ is a basis for $(X, \mathcal{T}_X)$ then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology of $Y \subset X$.*

Topological Bases

DEFN. A **topological basis** for a set X is a collection $\mathcal{B} \subset \mathcal{P}(X)$ such that

I. $\mathfrak{B}$ covers X.
II. For all $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is a union of basis elements

DEFN. The topology **generated** by a topological basis $\mathcal{B} \subset \mathcal{P}(X)$ is the set of all unions of elements from $\mathcal{B}$. Equivalently, a set $U \subset X$ is open iff for each $x \in U$ there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

DEFN. A subset $\mathcal{B} \subset \mathcal{T}$ is a **basis for the topology** $\mathcal{T}$ if every open set $U \in \mathcal{T}$ is a union of elements from $\mathcal{B}$.

- If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}$ is a topological basis.
- If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}$ generates $\mathcal{T}$.
- If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.
- First countable
- Second countable

DEFN. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology **generated** by the subbasis is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

I. The collection of finite intersections of subbasis elements is a basis.

Connectedness

DEFN. A topological space $(X, \mathcal{T})$ is **disconnected** iff $X = U \sqcup V$ is a union of disjoint open sets $U, V \in \mathcal{T}$, otherwise it is **connected**.

I. A set $A \subset X$ is **connected** if the subspace topology $\mathcal{T}_A$ is connected.
II. Connectedness is preserved by set intersection.

DEFN. A subset $A \subset X$ is **disconnected** if it is disconnected in the subspace topology, that is, there exist open $U, V \in \mathcal{T}$ such that

I. (Cover) $A \subset (U \cup V)$
II. (Relatively Disjoint) $(U \cap V) \cap A = \varnothing$
III. (Nonempty) $U \cap A, V \cap A \neq \varnothing$

We say that $A \subset (U \cup V)$ is a **disconnection** of A in $(X, \mathcal{T})$. Any subset that *crosses* a disconnection is itself disconnected; more precisely,

Proposition 1. *Let $(U \cup V)$ disconnect A. If $B \subset A$ intersects both U and V, then $(U \cup V)$ disconnects B.*

Theorem 4. *$(X, \mathcal{T})$ is disconnected if and only if there is a continuous onto function $f : X \to \{0, 1\}$, where $\{0, 1\}$ has the discrete topology.*

Proof. If such a function exists, then X disconnects as $X = f^{-1}\{0\} \sqcup f^{-1}\{1\}$. Conversely, if $X = U \sqcup V$ is disconnected, the function $f(x) = \mathbb{1}_U(x)$ is onto, since A, B nonempty, and continuous, since $U = f^{-1}(1)$ and $V = f^{-1}(0)$ are open. □

Theorem 5. *The continuous image of a connected set is connected.*

Proof. If $f : X \to Y$ is continuous with disconnected image $f(X) = U \sqcup V$, then $X = f^{-1}(U) \sqcup f^{-1}(V)$ is disconnected by continuity. □

EXAMPLE 3. We explore the connectedness of familiar spaces:

I. The trivial topology is always connected.
II. The discrete topology on X is disconnected when $|X| > 1$.
III. The lower limit topology on $\mathbb{R}$ is disconnected.
IV. The cofinite topology on an infinite set is connected.

Theorem 6. *The closure of a connected set is connected.*

Proof. By Property 1, every neighborhood of each $x \in \overline{A}$ intersects A, so A must cross every disconnection of $\overline{A}$. Therefore, any disconnection of $\overline{A}$ also disconnects A, by Proposition 1. □

Theorem 7. *If connected subsets $\{A_\alpha\}_{\alpha \in \Lambda}$ of $(X, \mathcal{T})$ share a common point $b \in \bigcap_\alpha A_\alpha$ then $A \equiv \bigcup_\alpha A_\alpha$ is connected.*

Proof. Suppose $(U \cup V)$ disconnects A, with $b \in U$. Then $(A \cap V)$ is nonempty, so some A_α intersects V. But $b \in A_\alpha$, so A_α also intersects U. Therefore, $U \cap V$ disconnects A_α, a contradiction! □

Total Disconnectedness

DEFN. A topological space $(X, \mathcal{T})$ is **totally disconnected** iff the only nonempty connected sets are singletons.

I. The product of totally disconnected spaces is totally disconnected.

EXAMPLE 4. The lower limit topology $\mathbb{R}_\ell$ is totally disconnected. Recall that the only connected subsets of $\mathbb{R}$ are intervals and singletons; since intervals are disconnected in the strictly finer topology $\mathbb{R}_\ell$, only singletons are connected. Alternatively, notice that if $A \subset \mathbb{R}_\ell$ contains two points $a < b$, then $(-\infty, b) \cup [b, +\infty)$ disconnects A.

Path-Connectedness

DEFN. In a topological space $(X, \mathcal{T})$, a **path** from x to y is a continuous map $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$.

I. Paths $x \rightsquigarrow y$ and $y \rightsquigarrow z$ can be appended to form a path $x \rightsquigarrow z$.

DEFN. A topological space $(X, \mathcal{T})$ is **path-connected** if there is a path between every two points. A subset $A \subset X$ is path-connected if the subspace topology is path-connected.

Theorem 8. *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous. If $(X, \mathcal{T}_X)$ is path-connected, then so is $f(X) \subset Y$.*

Proof. The pushforward $f \circ \gamma : [a, b] \to f(X)$ of any path $\gamma : [a, b] \to X$ from x_1 to x_2 in X is a path from $f(x_1)$ to $f(x_2)$ in the image $f(X)$. □

Theorem 9. *If $(X, \mathcal{T})$ is path-connected, then it is connected.*

Proof. Let $X = U \cup V$ be a disconnection of $(X, \mathcal{T})$. Suppose $\gamma : [a, b] \to X$ is a path from some $u \in U$ to $v \in V$. Since $[a, b]$ is connected, $\gamma([a, b]) \subset X$ must be connected; but $U \cap \gamma([a, b])$ and $V \cap \gamma([a, b])$ disconnect $\gamma([a, b])$, a contradiction! □

EXAMPLE 5. The sets $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. If a homeomorphism $f : \mathbb{R} \to \mathbb{R}^2$ were to exist, then the restriction $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}^2 \backslash \{f(0)\}$ would also be a homeomorphism; but this is impossible, since the image is connected but the domain is not!

Compactness

Sequential Compactness

DEFN. A topological space $(X, \mathcal{T})$ is **sequentially compact** iff every sequence has a convergent subsequence.

I. Every sequentially compact subset $C \subset X$ is closed.
II. Sequential compactness is preserved by set intersection.
III. A closed subset of a sequentially compact space is sequentially compact.

EXAMPLE 6. In the discrete metric, all sets are closed and bounded but fail to be sequentially compact.

Theorem 10. *The image $f(X)$ of a sequentially compact set X under a continuous map $f : X \to Y$ is sequentially compact.*

Proof. The preimage $(x_n) \subset X$ of any sequence $(f(x_n)) \subset f(X)$ has a convergent subsequence (x_{n_k}). By continuity, the image $(f(x_{n_k})) \subset f(X)$ converges. □

Corollary 2. *If $(X, \mathcal{T})$ is sequentially compact and $f : X \to Y$ continuous, then f is a closed map.*

Proof. Since X is sequentially compact, each closed $C \subset X$ is sequentially compact. Then, $f(X)$ is sequentially compact in Y, therefore closed. □

EXAMPLE 7. $\mathcal{C}([0, 1] \to \mathbb{R})$ with metric induced by $\|\cdot\|_\infty$ is unbounded, so fails to be sequentially compact.

Sequentially compact space is separable, totally bounded, complete

Compactness

DEFN. An **open cover** of a topological space $(X, \mathcal{T})$ or subset $A \subset X$ is a collection $\{U_\alpha\}_{\alpha\in I} \subset \mathcal{T}$ of open sets with $A \subset \bigcup_{\alpha\in I} U_\alpha$. A **subcover** is any subcollection $\{U_\alpha\}_{\alpha\in J}$ for $J \subset I$.

DEFN. A topological space $(X, \mathcal{T})$ or subset $A \subset X$ is **compact** iff every open cover has a finite subcover.

I. A finite union of compact sets is compact.
II. A subset is compact if and only if its subspace topology is compact.

Property 4, (Compactness in Subspaces)**.** *In a topological space $(Y, \mathcal{T}_Y)$, a set $A \subset X \subset Y$ is compact in the subspace topology $(X, \mathcal{T}_X)$ of X if and only if A is compact in the whole space $(Y, \mathcal{T}_Y)$.*

Proof. Every open cover of $A \subset Y$ by open sets $\{V_\alpha\}_{\alpha\in I}$ in Y corresponds to an open cover $\{U_\alpha = V_\alpha \cap X\}_{\alpha\in I}$ of $X \cap A$ by open sets in the subspace topology and vice-versa. Since $A \subset X$, we have $X \cap A = A$, so a finite subcover for one immediately gives a finite subcover for the other. □

EXAMPLE 8. The intersection of compact sets is not necessarily compact. Start with the discrete topolgoy on $\mathbb{N}$ and add two points x_1, x_2. Declare that the only open sets containing the new points are $\mathbb{N}\cup\{x_1\}$, $\mathbb{N}\cup\{x_2\}$, and $\mathbb{N} \cup \{x_1, x_2\}$. Then $\mathbb{N} \cup \{x_1\}$ and $\mathbb{N} \cup \{x_2\}$ are compact with non-compact intersection $\mathbb{N}$!

EXAMPLE 9. Verifying compactness for some familiar spaces.

I. Any topology with finitely many open sets is compact, including any topology on a finite set or the trivial topology on any set.
II. The discrete topology on an infinite set is never compact, because the singletons $\{\{x\}\}_{x \in X}$ cover X.
III. The reals $\mathbb{R}$ are not compact; consider any cover by bounded intervals.
IV. The lower-limit topology $\mathbb{R}_\ell$ fails to be compact for the same reason as $\mathbb{R}$.

Property 5. *A closed subset $A \subset X$ of a compact space $(X, \mathcal{T})$ is compact.*

Proof. Since A is closed, $X \setminus A$ together with any open cover of A form an open cover of X. Compactness gives a finite subcover of X, so also A. □

EXAMPLE 10. A compact set need not be closed. Consider the cofinite topology on $\mathbb{Z}$. The countable set $2\mathbb{Z} \subset \mathbb{Z}$ is not closed. However, $2\mathbb{Z}$ is compact, since any member $U \in \mathcal{U}$ of an open cover for $\mathbb{Z}$ ignores only finitely many elements.

EXAMPLE 11. The cofinite topology is always compact. Each member U_β of an open cover for X excludes only finitely many points from X. Cover the remaining points with finitely many more sets from the cover.

EXAMPLE 12. The cocountable topology is not compact over any infinite set X. If X is countable, the cocountable topology is discrete, so non-compact. For uncountable X, every countable subset $A \subset X$ is closed, with discrete subspace topology. Compactness of X would imply that A is compact, a contradiction of Property 5!

Theorem 11. *The continuous image of a compact set is compact.*

Proof. Let X compact and $f : X \to Y$ continuous. If $\{V_\alpha\}_{\alpha \in I} \subset \mathcal{T}_Y$ is an open cover of $f(X) \subset Y$, then $\{f^{-1}(V_\alpha)\}_{\alpha \in I}$ is an open cover of X, which must have a finite subcover $\{f^{-1}(V_{\alpha_k})\}_{k=1}^{n}$ by continuity. The corresponding sets $\{V_{\alpha_k}\}_{k=1}^{\infty}$ form a finite subcover of $f(X)$. □

Property 6, (Compactness in Hausdorff Spaces)**.** *Let $(X, \mathcal{T})$ be Hausdorff.*

I. *Every compact set is closed. (Theorem 15).*
II. *An arbitrary intersection of compact sets is compact. (Exercise 1)*

Continuous Functions on Compact Spaces

Theorem 12. *If $(X, \mathcal{T})$ is compact, then every continuous $f : X \to \mathbb{R}$ realizes its supremum, that is, there exists $z \in X$ such that $f(z) = \sup_{x \in X} f(x)$.*

Proof. By Theorem 11, the continuous image $f(X) \subset \mathbb{R}$ is compact in $\mathbb{R}$, so closed and bounded by Theorem 23. Thus $f(X)$ contains its supremum. □

Theorem 13. *If X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Proof. (TODO) Since f is a bijection, f^{-1} exists. It remains to show continuity. We begin by showing that f is a closed map. □

Why Compactness?

REMARK 1, (*Local-to-Global Property*, Jänich 1984, §8). Properties that hold locally in a compact space can be extended globally in the following way. Let $(X, \mathfrak{T})$ be compact and P be a property that the open subsets of X may or may not have, but which is preserved under finite unions. Then, if X has this property *locally*, that is, every point has a neighborhood with property P, then X itself has property P, since X is covered by a finite union of open neighborhoods with property P.

EXAMPLE 13. Simple examples of the local-to-global property.

I. If X is compact and $f : X \to \mathbb{R}$ is locally bounded (continuous, for example), then f is bounded.
II. Let X be compact and suppose $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly. Then the sequence converges uniformly on all of X.
III. Let X be compact and $\{U_\alpha\}_{\alpha\in I}$ be a locally finite cover (each point has a neighborhood intersecting only finitely many covering sets). Then the cover is finite.
IV. A vector field on a compact manifold without boundary is globally integrable!

Separation Axioms

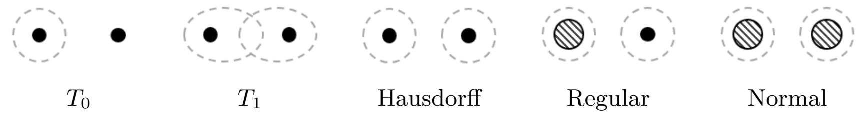


Figure 1: Points are dots; open sets are dashed; closed sets are hatched.

Axiom T_1

DEFN. A topological space $(X, \mathcal{T})$ is $\boldsymbol{T_1}$ iff iff for any distinct $x, y \in X$, there is a neighborhood of x not containing y (and vice versa).

I. A topological space is T_1 if and only if singletons are closed.

EXAMPLE 14. The following topological spaces are not T_1 spaces:

I. The trivial topology over any set.
II. The topology generated by the concentric circles $\{\|x\|_2 = r\} \subset \mathbb{R}^2$.
III. $\mathbb{R}$ with open sets $\mathcal{T} = \{\varnothing, \mathbb{R}\} \cup \mathcal{P}(\mathbb{Z})$. No two $x, y \notin \mathbb{Z}$ are separable.

Axiom T_2 (Hausdorff)

DEFN. A topological space $(X, \mathcal{T})$ is **Hausdorff** iff distinct points $x \neq y$ can always be separated by disjoint open neighborhoods $U_x, U_y \in \mathcal{T}$.

I. The continuous preimage of a Hausdorff space is Hausdorff.
II. Homeomorphisms preserve Hausdorff separability.
III. Sequences in a Hausdorff space have at most one limit.

EXAMPLE 15. The cofinite topology on $\mathbb{Z}$ is T_1 but not Hausdorff. Distinct $x, y \in \mathbb{Z}$ cannot have disjoint open neighborhoods $U_x, V_y \in \mathcal{T}_{\mathbb{Z}}$; otherwise, $U \cap V = \varnothing$ and $\mathbb{Z} \setminus U$ contains the infinite set V, contradicting that U is cofinite!

EXAMPLE 16. A topological space with unique limits is not necessarily Hausdorff. The cocountable topology on $\mathbb{R}$ is not Hausdorff, since all open sets intersect, but limits are still unique, since every convergent sequence is eventually constant.

Theorem 14. *If a first-countable space has unique limits, it is Hausdorff.*

Proof. It is enough to show that a first-countable non-Hausdorff space fails to have unique limits. Suppose $x, y \in X$ cannot be separated by disjoint neighborhoods. Let $U_1 \supset U_2 \supset \cdots \in \mathcal{N}_{\mathcal{T}}(x)$ and $V_1 \supset V_2 \supset \cdots \in \mathcal{N}_{\mathcal{T}}(y)$ be decreasing countable local bases for x and y, respectively. By assumption, we can pick $z_n \in U_n \cap V_n$ for all $n \in \mathbb{N}$. The sequence (z_n) converges to both x and y, since each neighborhood of either point contains a tail of the sequence. □

Theorem 15. *If $(X, \mathcal{T})$ is Hausdorff, then every compact set is closed.*

Proof. We show $X \setminus A$ is open for any compact $A \subset X$. Fix $y \in X \setminus A$. Separate y from each $x \in A$ by disjoint open sets $U_x, V_x \in \mathcal{T}$. By compactness, the open cover $\{V_x\}_{x \in A}$ of A has a finite subcover $\{V_{x_1}, \dots, V_{x_n}\}$. Thus $\cap_{k=1}^n U_k \subset X \setminus A$ is an open neighborhood of y, disjoint from A. □

Example 17. (TODO) The cocountable topology on $\mathbb{R}$ is not Hausdorff, since all open sets intersect, but compact sets must be finite, therefore closed.

Exercise 1. In a Hausdorff space $(X, \mathcal{T})$, any arbitrary intersection of compact sets is compact.

Regularity

Defn. A topological space $(X, \mathcal{T})$ is **regular** iff for all closed $C \subset X$ and $x \notin C$, there exist disjoint open $U, V \in T$ such that $C \subset U$ and $x \in V$.

Example 18. The space $(\mathbb{R}, \mathcal{T})$ where $\mathcal{T} \equiv \mathcal{T}_{Euc} \cup \{(a,b) \setminus \mathbb{Q} \mid a, b \in \mathbb{R}\}$ is Hausdorff, being a refinement of $\mathbb{R}$, but not regular. The closed set $\mathbb{Q}$ is not separable from any $x \notin Q$ by disjoint open sets.

Theorem 16. *Every compact Hausdorff space $(X, \mathcal{T})$ is regular.*

Proof. Let $C \subset X$ be closed and $x \notin C$. Using the Hausdorff property, separate each $y \in C$ from x with disjoint open sets $y \in U_y$, $x \in V_y$. As a closed subset of a compact space, C is compact by Property 5, so the open cover $\{U_y\}_{y \in C}$ of C has a finite subcover $\{U_k\}_{k=1}^n$. Therefore, $V = \bigcap_{k=1}^n V_k$ and $U = \bigcup_{k=1}^n U_k$ are disjoint, open, and separate x from C. □

Normality

Defn. A topological space $(X, \mathcal{T})$ is **normal** if for any disjoint closed sets $A, B \subset X$, there are disjoint open sets $U, V \in \mathcal{T}$ with $A \subset U$ and $B \subset V$.

Theorem 17. *Proofs for the following facts can be found in Munkres [2].*

I. *Every second-countable regular space is normal.*
II. *Every metrizable space is normal.*
III. *Every compact Hausdorff space is normal.*

Defn. Subsets $A, B \subset X$ of a topological space $(X, \mathcal{T})$ can be **separated by a continuous function** there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Theorem 18, (Urysohn Lemma)**.** *In a normal topological space $(X, \mathcal{T})$, any two disjoint closed subsets can be separated by a continuous function.*

Proof. For a proof, see (Munkres 2000, §33). □

Defn. A topological space $(X, \mathcal{T})$ is **completely regular** or $\boldsymbol{T_{3\frac{1}{2}}}$ if it is T_1 and additionally every closed set can be separated from any point by a continuous function.

Every completely regular space is regular, and by Urysohn's lemma every normal space is completely regular.

Countability Axioms

DEFN. In a topological space $(X, \mathcal{T})$, denote the open neighborhoods of a point $x \in X$ by $\mathcal{N}_\mathcal{T}(x) = \{U \in \mathcal{T} \mid x \in U\}$.

DEFN. In a topological space $(X, \mathcal{T})$, a **local base** at $x \in X$ is a collection $\mathcal{B}_x \subset \mathcal{N}_\mathcal{T}(x)$ of neighborhoods such that for any $U \in \mathcal{N}_\mathcal{T}(x)$, there is $B \in \mathcal{B}_x$ such that $B \subset U$.

DEFN. A topological space $(X, \mathcal{T})$ is **first countable** iff X has a countable local base at every point, and **second countable** iff it has a countable basis.

I. Every second countable space is first countable.

First-Countability

Property 7. *If $(X, \mathcal{T})$ is first-countable, then there is a* decreasing *countable local base, $B_1 \supset B_2 \supset \cdots \in \mathcal{N}_\mathcal{T}(x)$ at each point.*

Proof. For any countable local base $\{B_n\}_{n=1}^\infty \subset \mathcal{N}_\mathcal{T}(x)$, the decreasing sequence $U_n = \bigcap_{k=1}^n B_n \subset B_n$ is also a countable local base. □

DEFN. A subset $A \subset X$ is **dense** in topological space $(X, \mathcal{T})$ iff $\overline{A} = X$. The space $(X, \mathcal{T})$ is **separable** iff there is a countable dense subset.

Theorem 19. *Let $(X, \mathcal{T})$ be first-countable and $A \subset X$. For every $x \in \overline{A}$, there is a sequence $x_1, x_2, \ldots \in A$ with $x_n \to x$.*

Proof. Take a decreasing countable local base $B_1 \supset B_2 \supset \cdots \in \mathcal{N}_\mathcal{T}(x)$ at $x \in \overline{A}$. By definition of closure, every neighborhood of x intersects A, so we can find $x_n \in (B_n \cap A)$. Then $x_n \to x$, since each neighborhood $U \subset \mathcal{N}_\mathcal{T}(x)$ has $U_n \subset B_n \subset U$ for some $n \in \mathbb{N}$. □

Corollary 3. *Let the topological space $(X, \mathcal{T})$ be first-countable. A set $C \subset X$ is closed if and only if it is sequentially closed.*

Corollary 4. *Every point in a first-countable topological space $(X, \mathcal{T})$ is the limit of some sequence.*

Second-Countability

Theorem 20. *Every second-countable space $(X, \mathcal{T})$ is separable.*

Proof. Let $\mathcal{B} = \{B_1, B_2, \ldots\} \subset \mathcal{T}$ be a countable basis. Choose $x_n \in B_n$ arbitrarily and let $A = \{x_n\}_{n \in \mathbb{N}}$. Now, every open neighborhood $U \in \mathcal{N}_\mathcal{T}(x)$ of a point $x \in X$ must contain a basis set $x_n \in B_n \subset U$, so $\overline{A} = X$ and A is dense! □

Product Spaces

Product Topology on $X \times Y$

Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, there is a natural way to define a topology on the product space $X \times Y$.

DEFN. For topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the **product topology** on $X \times Y$ is the topology $\mathcal{T}_{X \times Y}$ generated by the basis $\mathcal{B}_{\text{box}}$ given by

$$\mathcal{B}_{\text{box}} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

Property 8. *The collection $\mathcal{S} = \{\pi_X^{-1}(U) \mid U \in T_X\} \cup \{\pi_Y^{-1}(V) \mid V \in T_Y\}$ is a subbasis for the product topology on $X \times Y$.*

Theorem 21. *If $X \times Y$ is compact, then X is compact.*

Proof. The projection map is continuous, so $\pi_X(X \times Y) = X$ is compact.□

Property 9. *Which properties are preserved under the operation of taking products?*

- *Sequential compactness. YES!*
- *Connectedness. YES!*
- *Hausdorffness. YES!*
- *T_1ness. YES!*

Topology on Arbitrary Products

It is also possible to define a topology on arbitrary product spaces,

Box Topology, from basis $\mathcal{B}_{\text{box}} = \{U_1 \times U_2 \times \cdots \mid U_k \in \mathcal{T}_k\}$.
Product Topology, from subbasis $\mathcal{S} = \bigcup_{k \in \mathbb{N}} \{\pi_{X_k}^{-1}(U_k) \mid U_k \in \mathcal{T}_k\}$.

First, we must be more precise about what we mean by *infinite products*.

DEFN. Let J be an index set. Given a set X, a ***J*-tuple** of elements of X is a function $x : J \to X$, denoted $(x_\alpha)_{\alpha \in J}$.

DEFN. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets and let $X = \bigcup_{\alpha \in J} A_\alpha$. The **cartesian product** of $\{A_\alpha\}_{\alpha \in J}$, denoted by $\prod_{\alpha \in J} A_\alpha$ is defined to be the set of all J-tuples of elements of X such that $x_\alpha \in A_\alpha$ for all $\alpha \in J$.

Quotient Spaces

Equivalence Classes

NOTATION 1. For an equivalence relation $\sim$ on a set X, let $X/\sim$ denote the set of equivalence classes and $[x] \in X/\sim$ denote the equivalence class of x. Let $\pi : x \mapsto [x]$ be the canonical projection from X to $X/\sim$.

DEFN. The **quotient topology** on $X/\sim$ is the finest topology making π a continuous map, that is, $U \subset X/\sim$ is open in the quotient topology iff $\pi^{-1}(U)$ is open in X.

Quotient Maps

DEFN. A surjective map $p : X \to Y$ between topological spaces $(X, \mathcal{T}_x)$, $(Y, \mathcal{T}_Y)$ is a **quotient map** provided that $U \in \mathcal{T}_Y \iff p^{-1}(U) \in \mathcal{T}_X$.

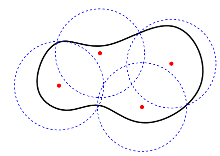
Metric Spaces

DEFN. A **metric space** (X, d) is a set with a distance $d : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$,

I. $d(x, y) = 0$
II. $d(x, y) = d(y, x)$
III. $d(x, z) \le d(x, y) + d(y, z)$

DEFN. The **metric topology** on (X, d) is that induced by the open balls $B_\varepsilon(x_0) \equiv \{x \in X \mid d(x, x_0) < \varepsilon\}$.

Total Boundedness



DEFN. A subset $A \subset X$ of metric space (X, d) is an **ε-net** iff every point is ε-close to A, that is, $\{B(a, \varepsilon)\}_{a \in A}$ is an open cover of X.

DEFN. A metric space is **totally bounded** if it can be covered by finitely many arbitrarily small open sets, that is, a finite ε-net exists for every $\varepsilon > 0$.

Theorem 22. *A metric space (X, d) is sequentially compact if and only if it is complete and totally bounded.*

Proof. Adapted from (Hunter and Nachtergaele 2001).

$\underline{\Longleftarrow}$ A complete, totally bounded metric space is sequentially compact.

For each $n \in N$, let $F_n \subset X$ be a finite $1/n$-net. Some open ball B_1 from the coarsest net F_1 must contain infinitely many terms of the sequence. Similarly, some $B_2 \subset B_1$ from F_2 contains infinitely many terms. Proceeding inductively, we find a sequence $B_1, B_2, \dots$ of shrinking open balls each containing infinitely many terms of (x_n). Choosing one point from each yields a convergent subsequence.

$\underline{\Longrightarrow}$ A sequentially compact metric space is totally bounded.

If no finite ε-net exists, then for each finite $F \subset X$ there is $y \in X$ with $d(y, F) > \varepsilon$. Let $x_0 \in X$ be any point. For any $n \in \mathbb{N}$, choose $x_n \in X$ with $d(x_n, \{x_k\}_{k=1}^{n-1}) > \varepsilon$. Any subsequence of (x_n) contains only points of mutual distance $\ge \varepsilon$ from one another. Such a sequence is not Cauchy, so fails to converge, violating sequential compactness.

$\underline{\Longrightarrow}$ A sequentially compact metric space is complete.

A Cauchy sequence converges to the limit of any convergent subsequence, guaranteed to exist by sequential compactness. □

Corollary 5. *A sequentially compact metric space (X, d) is separable.*

Proof (from [3]). By Theorem 22, (X, d) has a finite $(1/n)$-net A_n for all $n \in \mathbb{N}$. The set $A = \bigcup_{n=1}^\infty A_n$ is a countable dense subset, by construction. □

Compactness in Metric Spaces

Theorem 23. *In a metric space, every compact set is closed and bounded.*

Proof. Metric spaces are Hausdorff, so by Theorem 15, compact sets are closed. For boundedness, note that the open balls $\{B_n(0)\}_{n=1}^\infty$ cover any set, and only finitely many are needed to cover a compact set. □

Compact $\implies$ Sequentially Compact

Lemma 1. *In a metric space (X, d), suppose the sequence $(x_n)_{n=1}^\infty$ has no convergent subsequence. Then for all $x \in X$, there is $\varepsilon_x > 0$ such that $B(x, \varepsilon_x)$ contains only finitely many terms of the sequence.*

Proof. Suppose $x \in X$ exists such that for any $\varepsilon > 0$, $B(x, \varepsilon)$ contains infinitely many terms of the sequence. Let $n_0 = 0$. For $k \in \mathbb{N}$, pick $n_k > n_{k-1}$ with $x_{n_k} \in B(x, \frac{1}{k})$. Then $x_{n_k} \to x$ is a convergent subsequence! □

Theorem 24. *Every compact metric space (X, d) is sequentially compact.*

Proof. Suppose (X, d) is not sequentially compact, and take $(x_n)_{n=1}^\infty$ with no convergent subsequence. The sets $\{B(x, \varepsilon_x)\}_{x \in X}$ from Lemma 1 form an open cover of X. Since each $B(x, \varepsilon_x)$ contains only finitely many terms of $(x_n)_{n=1}^\infty$, there can be no finite subcover, and X fails to be compact. □

Sequentially Compact $\implies$ Compact

DEFN. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of the metric space (X, d). We say $\delta > 0$ is a **Lebesgue number** for $\mathcal{U}$ iff for all $x \in X$, there is $\alpha_x \in I$ such that $B(x, \delta) \subset U_{\alpha_x}$.

Lemma 2. *Suppose an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of metric space (X, d) has no positive Lebesgue number. Then there is a sequence $(x_n) \subset X$ such that for all $n \in \mathbb{N}$, the set $B(x_n, \frac{1}{n})$ is not entirely contained by any U_α, that is,*

$$\forall\, n \in N, \forall\, \alpha \in I, \exists\, x \in B(x_n, \tfrac{1}{n}) \text{ s.t. } x \notin U_\alpha \tag{1}$$

Proof. Negating the definition, $\forall\, \delta > 0, \exists\, x \in X, \forall\, \alpha \in I, B(x, \delta) \not\subset U_\alpha$. Construct the desired sequence by choosing x_n corresponding to $\delta_n = \frac{1}{n}$. □

Lemma 3, (Lebesgue Number Lemma)**.** *Every open cover of a sequentially compact metric space (X, d) has a positive Lebesgue number.*

Proof. When no positive Lebesgue number exists, the previous lemma gives a sequence (x_n) in X such that for all $n \in \mathbb{N}$, the set $B(x_n, \frac{1}{n})$ is not entirely contained by any U_α. By sequential compactness, there is a subsequence (x_{n_k}) converging to some $x \in X$, which lies in some member U_α of the open cover. Then for some $\varepsilon > 0$ and for k large enough we have $B(x_{n_k}, 1/n_k) \subset B(x, \varepsilon) \subset U_\alpha$, a contradiction! □

Theorem 25. *Every sequentially compact metric space (X, d) is compact.*

Proof. Let $\mathcal{U}$ be an open cover of (X, d). By Lemma 3, $\mathcal{U}$ has a positive Lebesgue number $\delta > 0$. By Theorem 22, (X, d) has a finite δ-net $F \subset X$, that is, for all $x \in X$, there is $x_0 \in F$ with $x \in B_\delta(x_0)$. Further, by definition of Lebesgue number, for all $x_0 \in F$ there is $\alpha_{x_0} \in I$ such that $B_\delta(x_0) \subset U_{\alpha_{x_0}}$. Together, we see that $\{U_{\alpha_{x_0}}\}_{x_0 \in F}$ is a finite subcover so X is compact. □

Continuity & Uniform Continuity in Metric Spaces

Theorem 26. *Let (X, d_X) and (Y, d_Y) be metric spaces. If (X, d_X) is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.*

Proof. Pick $\varepsilon > 0$. By continuity, for each $x \in X$ there is $\delta_x > 0$ such that $f(B(x, \delta_x)) \subset B(f(x), \varepsilon/2)$. The open balls $\{B(f(x), \delta_x/2)\}_{x \in X}$ cover the compact set X, so there is a finite subcover $\{B(f(z_k), \delta_{z_k}/2)\}_{k=1}^n$. Take $\delta = \min_k \delta_{z_k}$. For any $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, we have $x_1, x_2 \in B(z_k, \delta_{z_k})$ for some z_k, so $f(x_1), f(x_2) \in B(f(z_k), \varepsilon/2)$ differ by at most ε. □

Uniform Convergence

DEFN. Let $f_n : X \to Y$ be a sequence of functions from a set X to a metric space (Y, d). The sequence (f_n) **converges uniformly** to $f : X \to Y$ iff

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\ \forall\, n > N,\ \forall\, x \in X, \quad d(f_n(x), f(x)) < \varepsilon$$

Theorem 27, (Uniform Limits Preserve Continuity)**.** *Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space $(X, \mathfrak{T})$ to a metric space (Y, d). If $f_n \to f$ uniformly, then $f : X \to Y$ is continuous.*

Proof (from Munkres [2]). Let $V \subset Y$ be open. To show $f^{-1}(V) \subset X$ is open, it suffices to find for every $x_0 \in f^{-1}(V)$ an open neighborhood $x_0 \in U_{x_0} \in \mathfrak{T}$ such that $U_{x_0} \subset f^{-1}(V)$, that is, $f(U_{x_0}) \subset V$.

Let $y_0 = f(x_0)$. Choose $\varepsilon > 0$ so that $B(y_0, \varepsilon) \subset V$. By uniform convergence, choose $N \in \mathbb{N}$ such that for all $n > N$ and $x \in X$, $d(f_n(x), f(x)) < \varepsilon/3$. By continuity of f_N, choose $\delta > 0$ such that $f_N(B(x_0, \delta)) \subset B(f_N(x_0), \varepsilon/3)$. Then,

$$\begin{aligned} d(f(x), f_N(x)) &\le \varepsilon/3 && \text{(by choice of } N) \\ d(f_N(x), f_N(x_0)) &\le \varepsilon/3 && \text{(by choice of } U_{x_0}) \\ d(f_N(x_0), f(x_0)) &\le \varepsilon/3 && \text{(by choice of } N) \end{aligned}$$

By the triangle inequality, $d(f(x), f(x_0)) < \varepsilon$, so $f(U) \subset B(y_0, \varepsilon) \subset V$. ☐

Examples and Counterexamples

Trivial. Coarsest topology $\mathcal{T} = \{\varnothing, X\}$.

- Always connected and compact.

Discrete. Finest topology $\mathcal{T} = \mathcal{P}(X)$, all subsets are open.

- Hausdorff for any X.
- Totally disconnected when $|X| > 1$.
- Not compact over any infinite set, since singletons are open.

Cofinite. $\mathcal{T} = \{U \subset X \mid X \setminus U \text{ finite}\}$.

- $X = \mathbb{Z}$: T_1 but not Hausdorff
- Connected for any infinite set.
- Always compact.

Cocountable. $\mathcal{T} = \{U \subset X \mid X \setminus U \text{ countable}\}$ on an infinite set.

- Not Hausdorff because all open sets intersect.
- Connected because all open sets intersect.
- Convergent sequences are eventually constant; thus limits are unique.
- Not compact over any infinite set.

Lower Limit ($\mathbb{R}_\ell$). Topology on $\mathbb{R}$ generated by half-open intervals $[a, b)$.

- Explicitly, $\mathcal{T} = \{U \subset \mathbb{R} \mid \forall x \in U, \exists \varepsilon > 0, [x, x + \varepsilon) \subset U\}$.
- Disconnected, since $\mathbb{R}_\ell = (-\infty, 0) \cup [0, +\infty)$.
- Totally disconnected by Example 4.
- Not compact, since $\mathbb{R}_\ell$ is a refinement of $\mathbb{R}$.

Discrete Metric. $d(x, y) = \mathbb{1}_{x=y}$

- Generates the discrete topology.
- Every subset is closed, bounded but not sequentially compact.

Order Topology. For an ordered set X, consider the topology generated by the intervals (a, b).

References

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[2] James R Munkres. *Topology*. Prentice Hall, 2000.

[3] John K Hunter and Bruno Nachtergaele. *Applied analysis*. World Scientific Publishing Company, 2001.

[4] Sarah Bray. Lecture Notes in Topology. University of Michigan, MATH 490, Fall 2016. URL http://www-personal.umich.edu/~brays/teaching/490Fa16/.