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Tensor Algebra & Kronecker Products

August 5, 2019

These notes are almost entirely paraphrased from one of the references, with some extra details filled in by me where needed.

Many of the *products* we have in linear algebra are bilinear maps:

scalar	$V \times F \to V$	inner	$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
cross	$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$	matrix	$\mathbb{R}^{m\times k} \times \mathbb{R}^{k\times n} \to \mathbb{R}^{m\times n}$

Accordingly, we might think it important to require that any new product we define is also bilinear.

1 Background: Free Vector Space over a Set

Definition 1. The **free vector space** generated by a set X over field $\mathbb{F}$ is the set of functions $X \to F$ which are nonzero at only finitely many points,

$$\mathrm{Free}(X) = \left\{ f : X \to \mathbb{F} \mid f^{-1}\left(\mathbb{F} \setminus \{0\}\right) \text{ is finite} \right\}$$

The free vector space generated by X is a vector space when equipped with the standard operations of function addition and scalar multiplication:

$$(f+g)(x) = f(x) + g(x) \qquad (\alpha f)(x) = \alpha \cdot f(x)$$

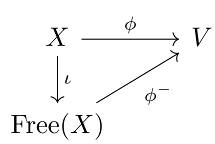
The **standard basis** for $\mathrm{Free}(X)$ is given by the indicator functions,

$$\mathrm{Free}(X) = \mathrm{span}\left\{ \mathbb{1}_a : X \to \{0,1\} \mid a \in X \right\}$$

Example 1. The free vector space on $[n]$ over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$.

The map $\iota : X \to \mathrm{Free}(X)$ mapping $(a \in X) \mapsto \mathbb{1}_a$ is a bijection. Accordingly, $\mathrm{Free}(x)$ may be regarded as the set of *formal* linear combinations of elements from X.

Remark 1. *The map ι is* universal *in the sense that if $\phi : X \to V$ is an arbitrary map from X to a vector space V, then there is a unique map ϕ^- such that the diagram below commutes:*



2 Direct Sums

For comparison to tensor products, the direct sum $V \oplus W$ of two vector spaces V, W by imposing a vector space structure on the Cartesian product $V \times W$, with addition and scaling defined by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$
$$\alpha(v_1, w_1) = (\alpha v_1, \alpha w_1)$$

So that expressions like $v + w = (v, w)$ are well-defined, it is convenient to identify $v \in V$ with the element $(v, 0) \in V \times W$ in the larger space.

Proposition 1. *If $\mathcal{B}_V = \{v_1, \dots, v_m\}$ and $\mathcal{B}_W = \{w_1, \dots, w_n\}$ are bases for V and W, respectively, then $\mathcal{B}_V \cup \mathcal{B}_W$ is a basis for $V \oplus W$. Consequently,*

$$\dim(V \oplus W) = \dim V + \dim W$$

Proposition 2. *Let $T \in L(V, V)$ and $U \in L(W, W)$ be linear operators. There is a unique linear operator $(T \oplus U) : (V \oplus W) \to (V \oplus W)$ such that*

$$(T \oplus U)(v \oplus w) = (Tv) \oplus (Uw) \qquad \text{for all } v \in V, w \in W$$

The matrix of $(T \oplus U)$ in any basis is block-diagonal, with blocks corresponding to the matrix representation of T and U respectively.

Proposition 3. *The direct sum satisfies the following useful properties:*

I. $\det(T \oplus U) = (\det T)(\det U)$
II. $\operatorname{tr}(T \oplus U) = \operatorname{tr}(T) \oplus \operatorname{tr}(U)$

3 Tensor Products

(wikipedia) The tensor product of V and W is the vector space generated by the symbols $v \otimes w$ with $v \in V, w \in W$, in which the relations of bilinearity are imposed for the product operation $\otimes$ and no other relations are assumed to hold. The tensor product space is thus the *freest* (or most general) such vector space, in the sense of having the *fewest* constraints.

Definition 2. The **tensor product** of two vector spaces V, W over $\mathbb{F}$ is a vector space $V \otimes W$ over $\mathbb{F}$ together with a bilinear map $V \times W \xrightarrow{\otimes} V \otimes W$ with the property that whenever $\mathcal{B}_V, \mathcal{B}_W$ are bases for V, W respectively, the set $\{v_i \otimes w_j \mid v_i \in \mathcal{B}_V, w_j \in \mathcal{B}_W\}$ gives a basis for $V \otimes W$.

The following theorem shows that the property in the definition need not be checked for all pairs of bases; just one pair is sufficient.

Theorem 1. *Let Y be a vector space and $\phi : V \times W \to Y$ be bilinear. Suppose there are bases $\mathcal{B}_V, \mathcal{B}_W$ for V, W respectively, such that $\phi(v \times w)$ is a basis for Y. Then the same holds for* any *pair of bases.*

3.1 Quotient Space Construction

Let V, W be vector spaces over $\mathbb{F}$. Following (Zakharevich 2015), our goal is to construct a vector space $V \otimes W$ such that for any vector space Z,

$$\mathcal{L}(V \otimes W, Z) \cong \left\{ \substack{\text{bilinear maps} \\ V \times W \to Z} \right\}$$

Step 1. Let $A = \operatorname{Free}(V \times W)$ be the free vector space over $\mathbb{F}$ on the product $V \times W$. We will demonstrate that

$$\mathcal{L}(A, Z) \cong \left\{ \substack{\text{functions} \\ V \times W \to Z} \right\}$$

For $v \in V, w \in W$, define the notation $v \otimes w \equiv \mathbb{1}_{(v,w)} \in A$. A typical vector has the form $f = \sum_{k=1}^{n} \alpha_k (v_k \otimes w_k) \in A$. By linearity, observe that each $T \in \mathcal{L}(A, Z)$ is determined uniquely by its values on the standard basis, exposing a bijection between $\mathcal{L}(A, Z)$ and (nonlinear) functions $V \times W \to Z$.

$$Tf = T\left(\sum_{k=1}^{n} \alpha_k (v_k \otimes w_k)\right) = \sum_{k=1}^{n} \alpha_k T(v_k \otimes w_k)$$

Step 2. Since $\mathcal{L}(A, Z)$ represents arbitrary functions, we will construct $V \oplus W$ by shrinking A so that $\mathcal{L}(V \times W, Z)$ represents only the bilinear maps $V \times W \to Z$. For example, if $T \in \mathcal{L}(V \otimes W, Z)$ is a linear map, the following proof of bilinearity in the first argument

$$\begin{aligned} T((v_1 + v_2) \otimes w) &= T((v_1 \otimes w) + (v_2 \otimes w)) \\ &= T(v_1 \otimes w) + T(v_2 \otimes w) \end{aligned}$$

requires that $(v_1 + v_2) \otimes w$ and $(v_1 \otimes w) + (v_2 \otimes w)$ refer to the same element in $V \otimes W$. Starting from A, we can enforce this equivalence implicitly by defining $V \otimes W$ to be the quotient space A/A_0, where

$$A_0 = \operatorname{span} \left\{ \begin{array}{r|l} (v_1 + v_2) \otimes w - (v_1 \otimes w) - (v_2 \otimes w) & \\ v \otimes (w_1 + w_2) - (v \otimes w_1) - (v \otimes w_2) & \alpha \in F \\ (\alpha v) \otimes w - \alpha (v \otimes w) & v \in V, w \in W \\ v \otimes (\alpha w) - \alpha (v \otimes w) & \end{array} \right\}$$

This choice was made so that each linear map $T \in \mathcal{L}(V \otimes W, Z)$ will satisfy[1]

$$\begin{aligned} T((v_1 + v_2) \otimes w) &= T(v_1 \otimes w) + T(v_2 \otimes w) \\ T(v \otimes (w_1 + w_2)) &= T(v \otimes w_1) + T(v \otimes w_2) \\ T((\alpha v) \otimes w) &= T(\alpha(v \otimes w)) \\ T(v \otimes (\alpha w)) &= T(\alpha(v \otimes w)) \end{aligned}$$

As pointed out by (Purbhoo 2012), since $A = \text{Free}(V \times W)$ is the space of formal linear combinations of (v, w) pairs, A_0 is the space of those linear combinations that can be simplified to the zero vector using bilinearity. Accordingly, A/A_0 reduces A to a space where two expressions are equal iff one can be simplified to the other using bilinearity.

3.2 Bilinearity, Linearity, and Universality

We have achieved our goal that $\mathcal{L}(V \otimes W, Z)$ is equivalent to the set of bilinear maps $V \times W \to Z$, for any vector space Z. Notice that we can easily translate between linear maps $\mathcal{L}(V \otimes W, Z)$ and bilinear maps $\mathcal{L}(V \times W, Z)$.

Linear to Bilinear. Given a linear map $T \in \mathcal{L}(V \oplus W, Z)$, the map $(v, w) \mapsto T(v \oplus w)$ is bilinear, since

I. the map $V \times W \mapsto V \otimes W$ given by $(v, w) \mapsto (v \otimes w)$ is bilinear
II. the composition of a bilinear map and a linear map is bilinear

Bilinear to Linear. Given a bilinear map $f : V \times W \to Z$, define a linear map $g : A \to Z$ by $g(v \oplus w) \mapsto f(v, w)$, which is well-defined since $(v \oplus w)$ form a basis for A. Moreover, bilinearity of f implies that g vanishes on the four types of spanning vectors which define A_0, so $g(A_0) = \{0\}$. Consequently, g descends to a well-defined linear map on A/A_0.

We have just proven the following *universality* theorem, which states that every bilinear map is the composition of a linear map with the tensor product.

Theorem 2. *(Universal Property) Let V, W, M be vector spaces over $\mathbb{F}$. For each bilinear map $\phi : V \times W \to M$, there is a unique linear transformation $\bar{\phi} : V \otimes W \to \mathcal{M}$ such that $\bar{\phi}(v \otimes w) = \phi(v, w)$ for all $v \in V, w \in W$. Moreover, every linear transformation in $\mathcal{L}(V \otimes W, M)$ arises in this way.*

More generally, every linear map $V_1 \otimes V_2 \otimes \cdots \otimes V_k \to M$ corresponds to a k-linear map $V_1 \times V_2 \times \cdots \times V_k \to M$.

3.3 Basis for the Tensor Product

Theorem 3. *Let $\{v_1, \ldots, v_m\} \subset V$ be a basis for V and $\{w_1, \ldots, w_n\} \subset W$. Then a basis for $V \otimes W$ is given by $\{v_i \otimes w_j \mid i \in [m], j \in [n]\}$.*

Proof. Notice that for any $v = \sum_{i=1}^m \alpha_i v_i$ and $w = \sum_{j=1}^n \beta_j w_j$,

$$\begin{aligned} v \otimes w = \left(\sum_{i=1}^m \alpha_i v_i \right) \otimes w &= \sum_{i=1}^m (\alpha_i v_i) \otimes w = \sum_{i=1}^m \alpha_i (v_i \otimes w) \\ &= \sum_{i=1}^m \alpha_i \left(v_i \otimes \sum_{j=1}^n \beta_j w_j \right) = \sum_{i=1}^m \sum_{j=1}^n \alpha_i \beta_j (v_i \otimes w_j) \end{aligned}$$

[1] Technically, $(v \otimes w) \in V \otimes W$ stands for the equivalence class of $v \otimes w = \mathbb{1}_{(v,w)} \in A$.

Therefore $V \otimes W = \text{span}\{v_i \otimes w_j \mid i \in [m], j \in [n]\}$. For independence, we will explicitly construct an isomorphism between $V \otimes W$ and an arbitrary (mn)-dimensional vector space Z with basis vectors z_{ij}. There is a unique linear map from $\varphi : Z \to V \oplus W$ defined by $\varphi(z_{ij}) = (v_i \otimes w_j)$. To show φ is an isomorphism, we manually construct an inverse ϕ^{-1}. Consider the linear map $A \to Z$ mapping

$$v \otimes w = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j (v_i \otimes w_j) \mapsto \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j z_{ij}$$

Notice that this map is zero on A_0, hence the map is well-defined on A/A_0. Moreover, it is surjective, since $v_i \otimes w_j \in A$, and inverse to ϕ. Hence ϕ is an isomorphism. ☐

Corollary 1. $\dim(V \otimes W) = \dim V \times \dim W$

Remark 2. *Although $V \otimes W$ is spanned by the* pure tensors *$v \otimes w$, but not every vector can be written in this form. There exist elements of the tensor product which are not pure tensors.*

4 Computations with Tensor Products

The construction above is axiomatic, and defines tensors in terms of the properties they should satisfy. In practice, it is useful to actually identify the space $V \otimes W$ and the bilinear map $V \times W \to V \otimes W$ with some familiar object. Note that we must specify *both* the vector space we are identifying with $V \otimes W$ *and* the product that we are using to make this identification.

4.1 Familiar Examples

Example 2, (Matrices)**.** Consider $V = \mathbb{F}^m$ and $W = \mathbb{F}^n$ To identify $V \otimes W$ with the space $\text{Mat}_{m\times n}(\mathbb{F})$ of matrices requires three steps:

- Define a vector space isomorphism $\phi : (\mathbb{F}^m \otimes \mathbb{F}^n) \to \text{Mat}_{m\times n}(\mathbb{F})$. Since these are finite-dimensional spaces with the same dimension, there is a natural isomorphism associated with any choice of bases.
- Define a bilinear map $\odot : \mathbb{F}^m \times \mathbb{F}^n \to \text{Mat}_{m\times n}(\mathbb{F})$ that explains how to construct a product element in $\text{Mat}_{m\times n}(\mathbb{F})$ from the two operands. This product should respect the isomorphism, $\phi(v \otimes w) = v \odot w$. For this example, one possible choice[2] is the outer product $v \odot w = vw^T$, meaning that identify vw^T with $v \otimes w$.
- Verify that $(\text{Mat}_{m\times n}(\mathbb{F}), \odot)$ is the tensor product of $\mathbb{F}^n$ and $\mathbb{F}^m$ by manually checking the basis condition in the definition. Indeed, choosing the standard basis $\{v_1, \dots, v_n\}$ for $\mathbb{F}^n$ and $\{w_1, \dots, w_m\}$ for $\mathbb{F}^m$, it is clear that the matrices $\{v_i w_j^T\}$ form a basis for $\text{Mat}_{m\times n}(\mathbb{F})$.

Example 3, (Polynomials)**.** Let $V = \mathbb{F}[x]$, the vector space of polynomials over one variable. Then $V \otimes V = \mathbb{F}[x, y]$ is the space of polynomials over two variables, with product $f(x) \odot g(y) = f(x)g(y)$. To verify, consider the basis $\mathcal{B} = \{1, x, x^2, \dots\}$ for V and observe that a basis for $\mathbb{F}[x, y]$ is

$$V \otimes V = \{x^a \otimes x^b \mid a, b \in \mathbb{N}\} \cong \{x^a y^b \mid a, b \in \mathbb{N}\}$$

[2] As (Purbhoo 2012) notes, there is more than one way to identify the tensor product of $\mathcal{F}^m$ and $\mathcal{F}^n$ with $\text{Mat}_{m\times n}$. They are all equivalent, in the same sense that $A \times B$ is equivalent to $B \times A$ for sets, but it is important to remember which choice was made.

Example 4. $V \otimes \mathbb{F} \cong V$, with product $v \otimes \alpha = \alpha v$. Choosing any basis $\{v_1, \ldots, v_n\}$ for V and the basis $\{1\}$ for $\mathbb{F}$, we see that $\{v_k \otimes 1\}_{k=1}^n \cong \{v_k\}_{k=1}$ recovers the same basis for V.

4.2 Revealing Examples

So far, our discussion of tensor products has not produced any unfamiliar objects. The next several examples are more interesting.

Example 5. Let V, W be finite dimensional vector spaces and consider the dual space $V^* = L(V, \mathbb{F})$. Then, as a generalization of Example 2, $V^* \otimes W = L(V, W)$, where $T \otimes w \in L(V, W)$ is defined to be the linear transformation $(f \otimes w)(v) = f(v) \cdot w$. If V and W are infinite-dimensional, then $V^* \otimes W = \{T \in L(V, W) \mid \dim(\operatorname{im} T) < \infty\}$ is the set of finite-rank linear transformations in $L(V, W)$. This follows from the fact that $T \otimes w$ has rank one, and a linear combination of such transformations must have finite rank.

Example 6. Tensor products work well to distinguish spaces with different *units*. For example, if V is the space of velocity vectors with units (m/s) and T is the space T of time measurements with units s, then $V \otimes T$ is the space of displacement vectors, with units (m). Although we can identify $V \cong \mathbb{R}^3$ and $T \cong \mathbb{R}$, this perspective emphasizes that V and T are fundamentally different.

Example 7. Let V, W, Z be vector spaces. Composition $(F, G) \mapsto F \circ G$ of linear maps is bilinear, giving rise to a linear map $\mathcal{L}(V, W) \otimes \mathcal{L}(W, Z) \to \mathcal{L}(V, Z)$.

4.3 Exotic Examples

4.4 Trace Functional

Let V be finite-dimensional. By Example 5, $V^* \otimes V \cong L(V \to V)$. Notice that the evaluation functional $(f, v) \mapsto f(v)$ from $V^* \times V \to \mathbb{F}$ is bilinear.

Definition 3. The **trace** functional on $L(V \to V)$ is the unique linear transformation $\operatorname{tr} : V^* \otimes V \to \mathbb{F}$ such that $\operatorname{tr}(f \otimes v) = f(v)$ for all $v \in V$.

Example 8, (Contraction)**.** Let $V_1, V_2, \ldots, V_k$ be f.d.v.s over $\mathbb{F}$. Suppose $V_i = V_j^*$ for some pair (i, j). Consider the products $V = \bigotimes_{r=1}^n V_r$ and $W = \bigotimes_{r \neq i,j} V_r$. Using the trace, a **contraction** with respect to (i, j) is the linear transformation $\operatorname{tr}_{ij} : V \to W$ satisfying

$$\operatorname{tr}_{ij}(v_1 \otimes \cdots \otimes v_k) = \operatorname{tr}(v_i \otimes v_j)(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes \cdots \otimes v_k)$$

5 Tensor Products & Linear Maps

5.1 Tensor Product of Linear Maps

Proposition 4. *Let $T \in L(V_1, V_2)$ and $U \in L(W_1, W_2)$ be linear maps. There is a unique linear map $T \otimes U \in L(V_1 \otimes W_1 \to V_2 \otimes W_2)$ such that*

$$(T \otimes U)(v \otimes w) = (Tv) \otimes (Uw) \quad \text{for all } v \in V, w \in W$$

Proof. Apply Theorem 2 to the bilinear map $(v, w) \mapsto (Tv) \otimes (Uw)$. □

5.2 Matrix Kronecker Products

Let $\mathcal{B}_V = \{v_1, \dots, v_m\} \in V$ and $\mathcal{B}_W = \{w_1, \dots, w_n\} \in W$ be bases for V, W respectively. Suppose $T \in L(V, V)$ has matrix $A \in \mathbb{F}^{n \times n}$.

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}_V \mapsto \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}_V$$

On the tensor product space, the map $(T \otimes I_W)(v \otimes w)$ applies A to v, leaving w untouched. With respect to the basis $v_i \otimes w_j$, this transformation is represented in matrix form by

$$\begin{bmatrix} a_{11}I & a_{12}I & \cdots & a_{1m}I \\ a_{21}I & a_{22}I & \cdots & a_{2m}I \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}I & a_{m2}I & \cdots & a_{mm}I \end{bmatrix}$$

5.2.1 Kronecker Products with the Identity

Example 9. For $m = 2$ and $n = 3$, the Kronecker product matrix for $T \otimes I_W$ is

$$A \otimes I = \left[\begin{array}{ccc|ccc} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} \\ \hline a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} \end{array}\right] \in \mathbb{R}^{6 \times 6}$$

Example 10. For $m = 2$ and $n = 3$, a linear map $U \in L(W, W)$ with matrix $B \in \mathbb{R}^{3 \times 3}$ acts on $V \otimes W$ according to the Kronecker product

$$I_2 \otimes B = \left[\begin{array}{ccc|ccc} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{array}\right] \in \mathbb{R}^{6 \times 6}$$

5.2.2 Properties of Kronecker Products

Proposition 5. *The Kronecker product has the following properties.*

I. $(A_1 \otimes I)(A_2 \otimes I) = (A_1 A_2) \otimes I$

II. $(I \otimes B_1)(I \otimes B_2) = I \otimes (B_1 B_2)$
III. $(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I) = (A \otimes B)$
IV. $(A \otimes B)(v \otimes w) = (Av) \otimes (Bw)$

Proposition 6. *The Kronecker product has the following properties.*

I. $(A \otimes B)^T = (A^T \otimes B^T)$
II. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$
III. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

Proposition 7. *(Trace and Determinant) The Kronecker product satisfies*

I. $\det(A \otimes B) = (\det A)^m (\det B)^n$
II. $\operatorname{tr}(A \otimes B) = (\operatorname{tr} A)(\operatorname{tr} B)$

Example 11. (Quantum Mechanics, [3]) In quantum mechanics, each degree of freedom in a system is associated with a Hilbert space. For example, a free particle in three dimensions has three dynamical degrees of freedom p_x, p_y, p_z, corresponding to the momentum. The eigenstate of the full Hamiltonian is obtained by the tensor product of momentum eigenstates in each direction,

$$|p_x, p_y, p_z\rangle = |p_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle$$

Example 12. An inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is a bilinear map, so by Theorem 2 there is a corresponding linear map $F \in L(V \otimes V, \mathbb{F})$ such that $\langle v_1, v_2 \rangle = F(v_1 \otimes v_2)$.

6 The Tensor Algebra

7 The Symmetric Tensor Algebra

The universal property for tensor products (Theorem 2) gives a natural correspondence between bilinear maps $V \times W \to M$ and linear maps $V \otimes W \to M$. By restricting our attention to linear maps with additional structure, it may be possible to view M is a quotient space of $V \otimes W$.

Definition 4. Let V be finite-dimensional over $\mathbb{F}$. Define $T^0(V) = \mathbb{F}$ and

$$T^k(V) \equiv V^{\otimes k} = V \otimes V \otimes \overset{k}{\cdots} \otimes V \quad \text{for any } k \in \mathbb{N}$$

Let $\mathcal{C}^k \preccurlyeq T^k(V)$ be the subspace spanned by all vectors

$$(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_k) - (x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_k)$$

The k^{th} **symmetric power** of V is the quotient space $\mathrm{Sym}^k(V) = T^k(V)/\mathcal{C}^k$. Elements of this space are written $x_1 x_2 \cdots x_k \cong \mathrm{Proj}_{\mathrm{Sym}^k(V)}(x_1 \otimes \cdots \otimes x_k)$.

Proposition 8. *The space* $\mathrm{Sym}^k(V)$ *can be thought of as the space of degree-*k *polynomials over* V*, with product given by polynomial multiplication with respect to any given choice of basis. In particular,* $\mathrm{Sym}^k(V)$ *has the following properties.*

I. *The map* $(x_1, \dots, x_k) \mapsto x_1 x_2 \cdots x_k$ *from* $V^k \mapsto \mathrm{Sym}^k(V)$ *is bilinear.*
II. $x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)} = x_1 x_2 \cdots x_k$ *for any permutation* σ*.*
III. *If* $\{v_1, \dots, v_n\}$ *form a basis for* V*, then any element of* $\mathrm{Sym}^k(V)$ *can be written as a polynomial of degree* k *in terms of the basis elements.*
IV. *There is a bilinear map* $\mathrm{Sym}^a(V) \times \mathrm{Sym}^b(V) \to \mathrm{Sym}^{a+b}(V)$ *sending* $(x_1 x_2 \cdots x_a, y_1 y_2 \cdots y_b) \mapsto x_1 x_2 \cdots x_a y_1 y_2 \cdots y_b$.

8 The Skew-Symmetric Tensor Algebra

Consider taking the tensor product $V \otimes V$ of V with itself. We can define a different type of tensor product by imposing the additional relation $v_1 \otimes v_2 = -v_2 \otimes v_1$. Define $A_1 = \text{span}\{A_0, (v_1 \otimes v_2) + (v_2 \otimes v_1)\}$, and write

$$V \wedge V = A/A_1$$

Pure tensors in $V \wedge V$ are written using the **wedge product** notation $v_1 \wedge v_2$ to emphasize that we have skew-symmetry.

Example 13, (Differential Forms)**.** Consider the vector space $V = \mathcal{C}(U \to \mathbb{R})$ of continuous, real valued functions on an open set $U \subset \mathbb{R}^d$. The module of 1-forms on U is the module

$$\text{span}\{f_1 x_1 + \cdots + f_n dx_n\}$$

and the module of k-forms is the wedge product of this module with itself k times. Skew-symmetry ensures $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$.

Example 14, (Wedge Powers)**.** If $\{v_1, \dots, v_n\}$ form a basis for V, then the skew-symmetric tensor product $V \wedge V$ is spanned by the vectors $v_i \wedge v_j$ with $i < j$. Write $\bigwedge^k V$ for the wedge product of V with itself k times. If $\dim V = n$, then $\dim \bigwedge^k V = \binom{\dim V}{k}$. In particular, $\dim \bigwedge^{\dim V} V = 1$.

Example 15, (Determinant)**.** Let $T : V \to V$ be linear. Then

$$\bigwedge^{\dim V} T : \left(\bigwedge^{\dim V} V \right) \to \left(\bigwedge^{\dim V} V \right)$$

Since the dimension of each of the vector spaces on the right is one, this is just multiplication by a scalar! We can define the determinant as simply $\det T \equiv \bigwedge^{\dim V} T$.

8.1 Exterior Product

Definition 5. The **exterior algebra** over a vector space V is the collection

$$\Lambda(V) \equiv \bigwedge^0 V \oplus \bigwedge^1 V \oplus \bigwedge^2 V \oplus \cdots$$

The exterior forms a ring under the wedge product operation. A vector space over $\mathbb{F}$ with the structure of a ring is called an **$\mathbb{F}$-algebra**.

Proposition 9. *If $\alpha \in \bigwedge^a V$ and $\beta \in \bigwedge^b V$, then $\alpha \wedge \beta = (-1)^{ab}(\beta \wedge \alpha)$.*

8.2 Linear Transformations

Theorem 4. *Let V, M be vector spaces over $\mathbb{F}$. For any k-linear alternating function $\phi : V^k \to M$, there is a unique $\overline{\phi} : \bigwedge^k V \to M$ such that*

$$\overline{\phi}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \phi(v_1, v_2, \dots, v_k) \qquad \text{for all } v_1, \dots, v_k \in V$$

Moreover, every linear map $\bigwedge^k V \to M$ arises in this manner.

Proposition 10. *There is a bilinear map $\bigwedge^a V \times \bigwedge^b V \mapsto \bigwedge^{a+b} V$ such that*

$$\left(\bigwedge_{i=1}^a v_i, \bigwedge_{j=1}^b w_j \right) \mapsto v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_b$$

8.3 Linear Independence

A square matrix has nonzero determinant if and only if its rows are linearly independent. The wedge product generalizes this idea.

Theorem 5. *A set of vectors* $\{v_1, \ldots, v_k\} \in V$ *is linearly independent if and only if* $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0 \in \bigwedge^k V$.

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