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Inequalities with sup and inf

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Infimum

Definition 1. We write $\inf A = \lambda$ when $\lambda \in \mathbb{R}$ is the **greatest lower bound** of $A \subset \mathbb{R}$, that is,

I. $\forall a \in A,\ \lambda \le a$
II. $\forall \varepsilon > 0, \exists a \in A,\ a < \lambda + \varepsilon$

$$\inf A \le \lambda \iff \forall \varepsilon > 0, \exists a \in A,\ a \le \lambda + \varepsilon$$

$\Longrightarrow$ Let $\varepsilon > 0$ and $\delta = \lambda - \inf A \ge 0$. Then there exists $a \in A$ such that $a \le \inf A + (\varepsilon + \delta) = \lambda + \varepsilon$.

$\Longleftarrow$ Assume $\inf A > \lambda$ and let $\varepsilon = \inf A - \lambda > 0$; then there exists $a \le \lambda + \frac{\varepsilon}{2} < \inf A$, a contradiction!

$$\inf A < \lambda \iff \exists a \in A,\ a < \lambda$$

$\Longrightarrow$ Suppose $\inf A < \lambda$. Choose $\varepsilon < \lambda - \inf A$. Then there exists $a \in A$ such that $a < \inf A + \varepsilon < \lambda$.

$\Longleftarrow$ Suppose $a \in A$ with $a < \lambda$. We can't have $\inf A \ge \lambda$, otherwise $a \ge \inf A \ge \lambda$!

$$\inf A \ge \lambda \iff \forall a \in A,\ a \ge \lambda$$

$\Longrightarrow$ If $\inf A \ge \lambda$, then each $a \ge \inf A \ge \lambda$.

$\Longleftarrow$ If λ is a lower bound, the *greatest* lower bound must be at least as large, so $\inf A \ge \lambda$.

$$\inf A > \lambda \iff \exists \varepsilon > 0, \forall a \in A,\ a \ge \lambda + \varepsilon$$

$\Longrightarrow$ Suppose $\inf A > \lambda$. Let $\varepsilon = \inf A - \lambda$. Then each $a \ge \inf A = \lambda + \varepsilon$.

$\Longleftarrow$ Suppose $\forall a \in A,\ a \ge \lambda + \varepsilon$ where $\varepsilon > 0$. Then by the above result, $\inf A \ge \lambda + \varepsilon > \lambda$.

Supremum

Definition 2. Similarly, we write $\sup A = \mu$ for the **least upper bound** of $A \subset \mathbb{R}$,

I. $\forall a \in A, a \le \mu$
II. $\forall \varepsilon > 0, \exists a \in A, a > \mu - \varepsilon$

$$\sup A \ge \lambda \iff \forall \varepsilon > 0, \exists a \in A,\ a \ge \lambda - \varepsilon$$

$\Longrightarrow$ Let $\varepsilon > 0$ and $\delta = \sup A - \lambda \ge 0$. Then there exists $a \in A$ such that $a \ge \sup A - (\varepsilon + \delta) = \lambda - \varepsilon$.

$\Longleftarrow$ Assume $\sup A < \lambda$ and let $\varepsilon = \lambda - \sup A > 0$; there exists $a \ge \lambda - \frac{\varepsilon}{2} > \sup A$, a contradiction!

$$\sup A > \lambda \iff \exists a \in A,\ a > \lambda$$

$\Longrightarrow$ Suppose $\sup A > \lambda$. Choose $\varepsilon < \sup A - \lambda$. Then there exists $a \in A$ such that $a > \sup A - \varepsilon > \lambda$.

$\Longleftarrow$ Suppose $a \in A$ with $a > \lambda$. We can't have $\sup A \le \lambda$, otherwise $a \le \sup A \le \lambda$!

$$\sup A \le \lambda \iff \forall a \in A,\ a \le \lambda$$

$\Longrightarrow$ If $\sup A \le \lambda$, then each $a \le \sup A \le \lambda$.

$\Longleftarrow$ If λ is an upper bound, the *least* upper bound can be no larger, so $\sup A \le \lambda$.

$$\sup A < \lambda \iff \exists \varepsilon > 0, \forall a \in A,\ a \le \lambda - \varepsilon$$

$\Longrightarrow$ Suppose $\sup A < \lambda$. Let $\varepsilon = \lambda - \sup A$. Then each $a \le \sup A = \lambda - \varepsilon$.

$\Longleftarrow$ Suppose $\forall a \in A,\ a \le \lambda - \varepsilon$ where $\varepsilon > 0$. Then by the above result, $\sup A \ge \lambda + \varepsilon > \lambda$.